4.1: Basic Counting Techniques

Question 1. Recall that a binary string is a string using only the symbols 0 and 1. How many binary strings of length 4 are there? Write them down in an organized way that shows that you have included all of them.

Inclusion-Exclusion Principle. Recall from Section 2.2: If A and B are finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

An immediate consequence is the so-called Addition Principle.

Addition Principle. Let A and B be finite sets such that $A \cap B = \emptyset$; i.e. A and B are disjoint sets. Then there are $|A| + |B|$ ways to choose an element from $A \cup B$; i.e.

$$|A \cup B| = |A| + |B|.$$

In counting problems, disjoint sets usually take the form of mutually exclusive options or cases. If a person has an "either/or" choice, or a problem reduces to *separate* cases, the addition principle is usually called for.

Example 1. Ray owns five bicycles and three cars. He can get to work using any one of these vehicles. How many different ways can he get to work?

Example 2. On a certain day, a restaurant is serving 10 vegetarian meal options, 5 pork meal options, and 4 chicken meal options. How many options for meals can you choose from?

Theorem 1. *Suppose that $A_1, A_2, \ldots, A_n$ are pairwise disjoint finite sets for some $n \geq 2$, that is, $A_i \cap A_j = \emptyset$ for all i and J with $i \neq j$. Then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|.$$

Proof. Left as an exercise. Use induction on n. $\square$

Multiplication Principle. Counting the number of elements in a rectangular grid is easy: you multiply the number of rows by the number of columns. We can always think of a Cartesian product $A \times B$ of two finite sets A and B as a grid, say, with the columns indexed by A and the rows indexed by B. Thus, the multiplication principle states that the number of elements in $A \times B$ is $|A| \cdot |B|$.

Example 3. Ray owns five bicycles and three cars. He plans to ride a bicycle to and from work, and then take one of his cars to go to a restaurant for dinner. How many different ways can he do this?

Example 4. On a certain day, a restaurant serves 5 appetizers, 10 main courses, and 4 desserts. How many options do you have for a three-course meal?

Theorem 2. *Suppose that $A_1, A_2, \ldots, A_n$ are a collection of finite sets for some $n \geq 2$. Then*

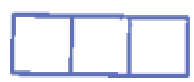
$$|A_1 \times A_2 \times \cdots \times A_n| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_n|.$$

Proof. Proof left as an exercise. Use induction on n. $\square$

Example 5. How many strings of length 3 can be formed from a 26-symbol alphabet?

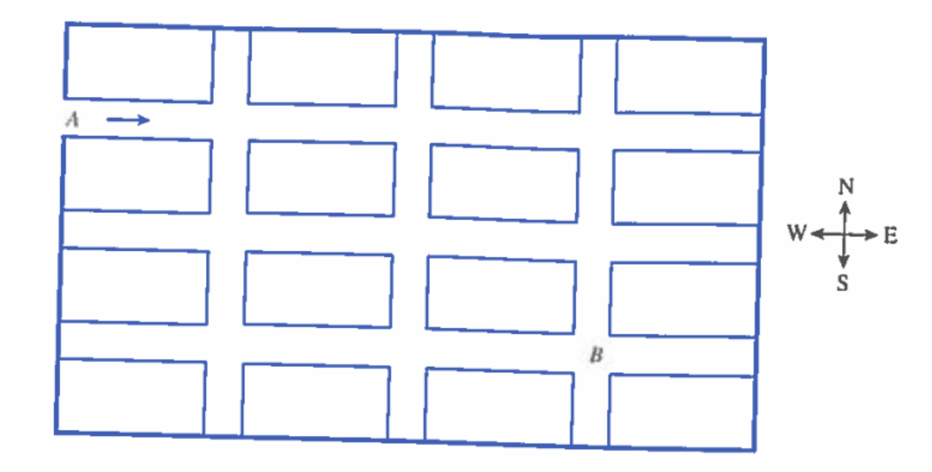
Example 6. How many different binary strings of length 24 are there?

Exercise 1. How many designs of the form



are possible, if each square must be either red, green, or blue, and no two adjacent squares may be the same color?

Exercise 2. The streets of a shopping district are laid out on a grid, as in the figure below. Suppose a customer enters the shopping district at point A and begins walking in the direction of the arrow. At each intersection, the customer chooses to go east or south, while taking as direct a path as possible to the bookstore at point B. How many different paths could the customer take?

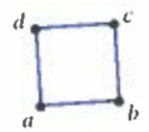


<u>Mixing Addition and Multiplication.</u> The addition and multiplication principles are pretty simple at face value. However, things get a little more tricky when you begin to combine them. The next few examples follow the same recipe: when a problem breaks up into disjoint cases, use multiplication to count each individual case, and then use addition to tally up the separate cases.

Example 7. How many (nonempty) strings of length at most length 3 can be formed from a 26-symbol alphabet?

Example 8. Illinois license plates used to consist of either three letters followed by three digits or two letters followed by four digits. How many such plates are possible?

Example 9. Using the four colors red, green, blue, and violet, how many different ways are there to color the vertices of the graph



so that no two adjacent vertices have the same color?

Homework. (Due Monday, December 3rd) Section 4.1: 2, 12, 16
Practice Problems. Section 4.1: 1, 3-7, 11, 13-14, 17-19, 21, **26, 27**